Int J Solids Structures Vol 15, pp 457–466
Pergamon Press Ltd, 1979 Printed in Great Britain

ELASTIC BUCKLING OF RECTANGULAR CLAMPED PLATES

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(Received 30 March 1978; in revised form 10 October 1978)

INTRODUCTION

The elastic buckling of flat rectangular plates, subjected to edge thrusts, and various types of edge restraint has been extensively studied and many numerical and analytical results for buckling loads are available (see, for example, Bulson[6] and Timoshenko[7]). However, for the particular case of a plate clamped on all four edges, the solution is rather more complicated, and the results are almost all restricted to uniaxially loaded plates. The first paper on the topic was by Taylor[1], who remarked on the complexity of the problem. He established a complete plate buckling theory for such plates, but was able only to give a single solution for a square plate with equal biaxial compression because of the complexity of the calculations. Subsequent researchers have solved more restricted problems, usually taking uniaxial loading only. Results have been obtained using various methods by Levy[2], Wittrick[3], Stowell[4] and El-Bayoumy[5]. Here we complete Taylor's theory, giving some terms which he omitted in the original paper, and applying a computer program to the numerical solution of his equations for a large range of plate parameters. In addition we consider the possibility of antisymmetrical modes of buckling which Taylor omitted for simplicity, and we show that by reversing the order in which the boundary conditions are applied, very similar equations are obtained to those for the symmetrical case. A complete set of formulae for researchers wishing to duplicate the results and a comprehensive table and graph of buckling loads are given.

Taylor developed a method to find solutions of the governing differential equation which satisfy zero displacement at the edges first. An infinite number of these can then be combined together in such a way that the remaining "clamped edge" condition (i.e. zero slope at the edges) is also satisfied, provided a certain relationship exists between the shape and size of the plate and the thrusts. Taylor considered only the displacements which are symmetrical with respect to both axes. His method can deal with a general range of loading cases and plate geometries. A computer program is used to perform the tedious calculations for various aspect ratios and load ratios.

THEORY

For convenience we now briefly recapitulate Taylor's theory.

The governing differential equation for elastic displacement w of a plane sheet subjected to stresses P_1 and P_2 parallel to rectangular axes ξ, η, is

$$\frac{\partial^4 w}{\partial \xi^4}+2\frac{\partial^4 w}{\partial \xi^2\partial \eta^2}+\frac{\partial^4 w}{\partial \eta^4}+\frac{P_1}{D}\frac{\partial^2 w}{\partial \xi^2}+\frac{P_2}{D}\frac{\partial^2 w}{\partial \eta^2}=0 \tag{1}$$

where $D=[Eh^2/12(1-\sigma^2)]$. Figure 1 shows the rectangular sheet loaded along its edges. h is plate thickness and σ Poisson's ratio.

Writing $\xi=2ax/\pi$, $\eta=2by/\pi$, the rectangle whose sides are $\xi=\pm a$, $\eta=\pm b$, is transformed into a square whose sides are $x=\pm\pi/2$, $y=\pm\pi/2$ and (1) becomes

$$\frac{1}{a^4}\frac{\partial^4 w}{\partial x^4}+\frac{2}{a^2b^2}\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{1}{b^4}\frac{\partial^4 w}{\partial y^4}+\frac{4P_1a^2}{D\pi^2}\left(\frac{1}{a^4}\frac{\partial^2 w}{\partial x^2}\right)+\frac{4P_2b^2}{D\pi^2}\left(\frac{1}{b^4}\frac{\partial^2 w}{\partial y^2}\right)=0. \tag{2}$$

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The problem is to find a solution of (2), other than $w = 0$, which satisfies the condition $w = 0$ at $x = \pm\pi/2$ and $y = \pm\pi/2$, and also the conditions $\partial w/\partial x = 0$ at $x = \pm\pi/2$ and $\partial w/\partial y = 0$ at $y = \pm\pi/2$. The method adopted is to find first solutions of the differential equation (2) which satisfy two of the four boundary conditions mentioned above. It is then shown that an infinite number of these can be combined together in such a way that the remaining two boundary conditions are also satisfied, provided a certain relationship exists between the size and shape of the plate and the thrusts P_1 and P_2.

Let us consider the symmetrical modes of buckling in which displacements are symmetrical with respect to both axes, so that x and y occur in w only as even functions.

The function $e^{\alpha y}\cos nx$ satisfies the condition $w = 0$ at $x = \pm\pi/2$, provided n is an odd integer and it also satisfies (2), provided

$$\frac{a^4}{b^4}\alpha^4 - 2n^2\alpha^2\frac{a^2}{b^2} + n^4 - \frac{4P_1a^2}{D\pi^2}n^2 + \frac{4P_2b^2a^4\alpha^2}{D\pi^2b^4} = 0. \tag{3}$$

By a suitable combination of the four terms of type $e^{\alpha y}\cos nx$, an even function is obtained which satisfies $w = 0$ at $y = \pm\pi/2$. These combinations will take four different forms, according to whether the roots of (3) are real, imaginary or complex.

Type 1

If the roots of (3), regarded as a quadratic in α^2, are positive, the appropriate form is as follows calling them α_n^2, β_n^2

$$w = \left(\cosh\alpha_n y\cosh\frac{\beta_n\pi}{2} - \cosh\beta_n y\cosh\frac{\alpha_n\pi}{2}\right)\cos nx.$$

Type 2

If the two roots of (3), regarded as a quadratic in α^2 are positive and negative, let them be $-\alpha_n^2$, β_n^2. The required term is

$$w = \left(\cosh\frac{\beta_n\pi}{2}\cos\alpha_n y - \cos\frac{\alpha_n\pi}{2}\cosh\beta_n y\right)\cos nx.$$

Type 3

If the four roots are pure imaginaries $\pm\alpha_n i$, $\pm\beta_n i$ the form is

$$w = \left(\cos\alpha_n y\cos\frac{\beta_n\pi}{2} - \cos\beta_n y\cos\frac{\alpha_n\pi}{2}\right)\cos nx.$$

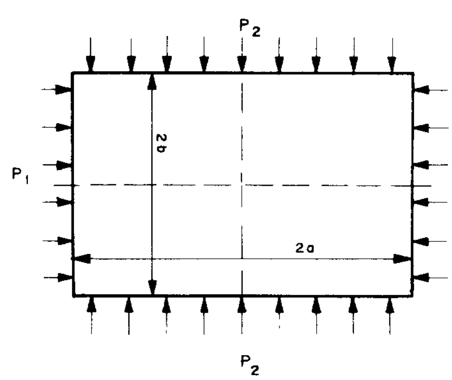

Fig. 1. Plate geometry.

Type 4

If the roots of (3) are complex they must be of the form $\pm\alpha_n \pm \beta_n i$ and the appropriate form for w is

$$w = \left(\cosh\alpha_n y\cos\beta_n y\sinh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2} - \sin\beta_n y\sinh\alpha_n y\cosh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}\right)\cos nx.$$

In the work which follows, only terms of type 1 will be referred to assuming that in applying the results terms of other types are substituted where necessary.

In order that a series of Type 1 may be capable of representing all possible values of w inside a square consistent with $w = 0$ at its edges, it must also be capable of representing arbitrary symmetrical distributions of $\partial w/\partial y$ along $y = \pm\pi/2$ and of $\partial w/\partial x$ along $x = \pm\pi/2$.

The single series $w = \sum_{n\text{ odd}} A_n\left(\cosh\alpha_n y\cosh\frac{\beta_n\pi}{2} - \cosh\beta_n y\cosh\frac{\alpha_n\pi}{2}\right)\cos nx$

is capable of representing any assigned distribution of $\partial w/\partial y$ along $y = \pm\pi/2$. A_n are unknown coefficients.

Similarly the series

$$w = \sum_{n\text{ odd}} B_n\left(\cosh\gamma_n x\cosh\frac{\delta_n\pi}{2} - \cosh\delta_n x\cosh\frac{\gamma_n\pi}{2}\right)\cos ny$$

is capable of representing any assigned distribution of $\partial w/\partial x$ along $x = \pm\pi/2$, and if $\pm\gamma_n$, $\pm\delta_n$ are the roots of the biquadratic

$$\frac{b^4}{a^4}\gamma^4 - 2n^2\gamma^2\frac{b^2}{a^2} + n^4 - \frac{4P_2b^2n^2}{D\pi^2} + \frac{4P_1a^2b^4}{D\pi^2a^4}\gamma^2 = 0 \tag{4}$$

each term of the series satisfies (2).

The two single series can be combined to form a double series

$$w = \sum A_n\left(\cosh\alpha_n y\cosh\frac{\beta_n\pi}{2} - \cosh\beta_n y\cosh\frac{\alpha_n\pi}{2}\right)\cos nx$$
$$+ B_n\left(\cosh\gamma_n x\cosh\frac{\delta_n\pi}{2} - \cosh\delta_n x\cosh\frac{\gamma_n\pi}{2}\right)\cos ny. \tag{5}$$

The determination of the actual values of the As and Bs in any given case would necessitate the solution of an infinite series of linear equations. If the two remaining boundary conditions (of $\partial w/\partial y$) are enforced, the solution of this series of equations would in general yield the result that all the As and Bs are zero, except if a certain relationship exists between the dimensions of the sheet and P_1 and P_2, namely that obtained by eliminating all the As and Bs from the system of linear equations.

To carry out the operations indicated above it is convenient to expand each term of the A and B series in (5) in a cosine series of even multiples of y. The coefficient of $\cos sy$ in the series so obtained is then equated to zero for each value (even) of s in order that $\partial w/\partial x = 0$ may be satisfied at all parts of the edges $x = \pm\pi/2$.

The necessary cosine series valid between $y = \pm\pi/2$ are

$$\cosh\alpha y = \frac{4\alpha}{\pi}\sinh\frac{\alpha\pi}{2}\left(\frac{1}{2\alpha^2} - \frac{\cos 2y}{\alpha^2+2^2} + \frac{\cos 4y}{\alpha^2+4^2} + \cdots\right) \tag{7}$$

$$\cos\alpha y = \frac{4\alpha}{\pi}\sin\frac{\alpha\pi}{2}\left(\frac{1}{2\alpha^2} - \frac{\cos 2y}{\alpha^2-2^2} + \frac{\cos 4y}{\alpha^2-4^2} - \cdots\right) \tag{8}$$

and if terms of Type 4 occur the expansions of $\cosh\beta y\cos\alpha y$ and $\sin\alpha y\sinh\beta y$ are also needed. (For their expansions, see Ref. [8].)

Inserting expansions of this type for $\cos ny$, $\cos\alpha_n y$ and $\cosh\beta_n y$ in the right hand side of (5) it is found that the condition $\partial w/\partial x = 0$ at $x = \pm\pi/2$ is satisfied if for every (even) value of s

$$\frac{4}{\pi}(-1)^{s/2}\sum_{n\ \mathrm{odd}}(-1)^{(n+1)/2}n(A_n a_{ns} - b_{ns}B_n) = 0 \tag{9}$$

where a_{ns}, b_{ns} take different forms according to whether the corresponding term is of Type 1, 2, 3 or 4. Their derivations are found in (8), while they are listed here as follows:

Type 1

$$a_{ns} = \frac{\alpha_n \sinh\frac{\alpha_n\pi}{2}\cosh\frac{\beta_n\pi}{2}}{\alpha_n^2 + s^2} - \frac{\beta_n \sinh\frac{\beta_n\pi}{2}\cosh\frac{\alpha_n\pi}{2}}{\beta_n^s + s^2}$$

$$b_{ns} = \frac{\gamma_n \sinh\frac{\gamma_n\pi}{2}\cosh\frac{\delta_n\pi}{2} - \delta_n \sinh\frac{\delta_n\pi}{2}\cosh\frac{\gamma_n\pi}{2}}{n^2 - s^2}.$$

Type 2

$$a_{ns} = \frac{\alpha_n \sin\frac{\alpha_n\pi}{2}\cosh\frac{\beta_n\pi}{2}}{\alpha_n^s - s^2} - \frac{\beta_n \sinh\frac{\beta_n\pi}{2}\cos\frac{\alpha_n\pi}{2}}{\beta_n^2 + s^2}$$

$$b_{ns} = \frac{-\left(\gamma_n \sin\frac{\gamma_n\pi}{2}\cosh\frac{\delta_n\pi}{2} + \delta_n \sinh\frac{\delta_n\pi}{2}\cos\frac{\gamma_n\pi}{2}\right)}{n^2 - s^2}.$$

Type 3

$$a_{ns} = \frac{\alpha_n \sin\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}}{\alpha_n^2 - s^2} - \frac{\beta_n \sin\frac{\beta_n\pi}{2}\cos\frac{\alpha_n\pi}{2}}{\beta_n^2 - s^2}$$

$$b_{ns} = \frac{-\gamma_n \sin\frac{\gamma_n\pi}{2}\cos\frac{\delta_n\pi}{2} + \delta_n \sin\frac{\delta_n\pi}{2}\cos\frac{\gamma_n\pi}{2}}{n^2 - s^2}.$$

Type 4

$$a_{ns} = \frac{1}{2}\left\{\sinh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2}\left[\frac{\sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}\alpha_n + (\beta_n - s)\cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2}}{\alpha_n^2 + (\beta_n - s)^2}\right.\right.$$
$$\left. + \frac{\alpha_n \sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2} + (\beta_n + s)\cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2}}{\alpha_n^2 + (\beta_n + s)^2}\right] - \cosh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}$$
$$\left[\frac{\cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2}\alpha_n - (\beta_n - s)\sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}}{\alpha_n^2 - (\beta_n - s)^2}\right.$$
$$\left.\left. + \frac{\alpha_n \cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2} - (\beta_n + s)\sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}}{\alpha_n^2 + (\beta_n + s)^2}\right]\right\}$$

$$b_{ns} = \frac{-\left(\delta_n \cosh\frac{\gamma_n\pi}{2}\sinh\frac{\gamma_n\pi}{2} + \gamma_n \cos\frac{\delta_n\pi}{2}\sin\frac{\delta_n\pi}{2}\right)}{n^2 - s^2}.$$

Permutating x and y the condition $\partial w/\partial y = 0$ at $y = \pm\pi/2$ leads to

$$\frac{4}{\pi}(-1)^{s/2}\sum(-1)^{(n+1)/2}n(c_{ns}B_n - d_{ns}A_n) = 0. \tag{10}$$

For instance, if α_n, β_n and also γ_n, δ_n are each derived from terms of Type 1

$$c_{ns} = \frac{\gamma_n \cosh\frac{\delta_n\pi}{2}\sinh\frac{\gamma_n\pi}{2}}{\gamma_n^2+s^2} - \frac{\delta_n \cosh\frac{\gamma_n\pi}{2}\sinh\frac{\delta_n\pi}{2}}{\delta_n^2+s^2}$$

$$d_{ns} = \frac{\left(\alpha_n \cosh\frac{\beta_n\pi}{2}\sinh\frac{\alpha_n\pi}{2} - \beta_n \sinh\frac{\beta_n\pi}{2}\cosh\frac{\alpha_n\pi}{2}\right)}{n^2-s^2}.$$

For Type 2, 3 and 4, the expressions for c_{ns}, d_{ns} can easily be obtained by permutating the corresponding a_{ns}, b_{ns}.

Eliminating the As and Bs between (9) and (10), we obtain the infinite determinant equation

$$\begin{vmatrix} a_{10} & b_{10} & a_{30} & b_{30} & \cdot & \cdot \\ d_{10} & c_{10} & d_{30} & c_{30} & \cdot & \cdot \\ a_{12} & b_{12} & a_{32} & b_{32} & \cdot & \cdot \\ d_{12} & c_{12} & d_{32} & c_{32} & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0. \qquad (11)$$

When a, b are fixed, the only variables remaining in Δ are P_1 and P_2, so that $\Delta = 0$ is a critical equation determining the special values of P_1 and P_2.

The same method could be applied to get anti-symmetrical modes whose displacements are symmetrical with respect to one axis and anti-symmetrical with respect to the other. Let us consider the case where anti-symmetry is in the x direction (the case when $a > b$).

The function $e^{\alpha y} \sin nx$ satisfies the condition $\partial w/\partial y = 0$ at $x = \pm\pi/2$ if n is odd and it also satisfies (2), provided (3) holds. It is worth noting that the condition $w = 0$ at $x = \pm\pi/2$ is not applied first since mathematical difficulty arises from the 'even n' in $e^{\alpha y} \sin nx$.

By combining the four terms of type $e^{\alpha y} \sin nx$, an odd function is obtained which satisfies $w = 0$ at $y = \pm\pi/2$. Again the roots of (3) can fall into four categories with their corresponding expressions for w. The four expressions are the same as those derived above for symmetrical modes, except that the term $\cos nx$ is replaced by $\sin nx$.

Consider Type 1 as an example.

$$\text{The single series is } w = \sum A_n\left(\cosh \alpha_n y \cosh\frac{\beta_n\pi}{2} - \cosh \beta_n y \cosh\frac{\alpha_n\pi}{2}\right)\sin nx. \qquad (12)$$

$$\text{Similarly the series } w = \sum B_n\left(\delta_n \cosh\frac{\delta_n\pi}{2}\sinh \gamma_n x - \gamma_n \cosh\frac{\gamma_n\pi}{2}\sinh \delta_n x\right)\cos ny \qquad (13)$$

is obtained which also satisfies the two boundary conditions and (2) if (4) holds.

Consider the double series

$$\sum A_n\left(\cosh \alpha_n y \cosh\frac{\beta_n\pi}{2} - \cosh \beta_n y \cosh\frac{\alpha_n\pi}{2}\right)\sin nx$$
$$+ B_n\left(\delta_n \cosh\frac{\delta_n\pi}{2}\sinh \gamma_n x - \gamma_n \cosh\frac{\gamma_n\pi}{2}\sinh \delta_n x\right)\cos ny. \qquad (14)$$

The two boundary conditions left unsatisfied are $\partial w/\partial y = 0$ at $y = \pm\pi/2$ and $w = 0$ at $x = \pm\pi/2$. Application of them leads to an infinite number of linear equations which will give trivial solutions unless As and Bs are eliminated. The process is similar to that explained above, except that sine series are needed for the expansions when $\partial w/\partial y = 0$ at $y = \pm\pi/2$ is applied. The necessary sine series valid between $x = \pm\pi/2$ are

$$\sin \alpha x = \frac{4}{\pi}\sin\frac{\alpha\pi}{2}\left(\frac{-2}{\alpha^2-2^2} + \frac{4}{\alpha^2-4^2} - \frac{6}{\alpha^2-6^2} + \cdots\right) \qquad (15)$$

$$\sinh \alpha x = \frac{4}{\pi} \sinh \frac{\alpha\pi}{2} \left(\frac{2}{\alpha^2 + 2^2} - \frac{4}{\alpha^2 + 4^2} + \frac{6}{\alpha^2 + 6^2} - \cdots \right) \tag{16}$$

and if Terms of Type 4 occur the expansions of $\cosh \alpha x \sin \beta x$ and $\sinh \alpha x \cos \beta x$ are also needed (see Ref. [8] for their expansions). Inserting expansions of this type for $\sin nx$, $\cosh \gamma_n x$ and $\cosh \beta_n x$ in the right hand side of (14), it is found that the condition $\partial w/\partial y = 0$ at $y = \pm \pi/2$ is satisfied if for even value of s ($s \neq 0$)

$$\frac{4}{\pi} s(-1)^{(s/2)+1} \sum (-1)^{(n-1)/2} [nB_n b_{ns} - A_n a_{ns}] = 0 \tag{17}$$

where a_{ns}, b_{ns} take different forms according to whether Type 1, 2, 3 or 4 occurs. Their derivations are again found in Ref. [8] while they are listed as follows:

Type 1

$$a_{ns} = \frac{\alpha_n \cosh \frac{\beta_n \pi}{2} \sinh \frac{\alpha_n \pi}{2} - \beta_n \cosh \frac{\alpha_n \pi}{2} \sinh \frac{\beta_n \pi}{2}}{n^2 - s^2}$$

$$b_{ns} = \frac{\gamma_n \cosh \frac{\gamma_n \pi}{2} \sinh \frac{\delta_n \pi}{2}}{\delta_n^2 + s^2} - \frac{\delta_n \cosh \frac{\delta_n \pi}{2} \sinh \frac{\gamma_n \pi}{2}}{\gamma_n^2 + s^2}$$

$w = 0$ at $x = \pi/2$ give similar equations in c_{ns}, d_{ns} for every even value of s (including $s = 0$) as

$$\frac{4}{\pi} (-1)^{s/2} \sum (-1)^{(n-1)/2} [A_n d_{ns} - nB_n c_{ns}] = 0. \tag{18}$$

The four pairs of expression for c_{ns}, d_{ns} are:

Type 1

$$c_{ns} = \frac{\gamma_n \cosh \frac{\gamma_n \pi}{2} \sinh \frac{\delta_n \pi}{2} - \delta_n \cosh \frac{\delta_n \pi}{2} \sinh \frac{\gamma_n \pi}{2}}{n^2 - s^2}$$

$$d_{ns} = \frac{\alpha_n \cosh \frac{\beta_n \pi}{2} \sinh \frac{\alpha_n \pi}{2}}{\alpha_n^2 + s^2} - \frac{\beta_n \sinh \frac{\beta_n \pi}{2} \cosh \frac{\alpha_n \pi}{2}}{\beta_n^2 + s^2}.$$

Type 2

$$c_{ns} = \frac{\gamma_n \cos \frac{\gamma_n \pi}{2} \sinh \frac{\delta_n \pi}{2} - \delta_n \cosh \frac{\delta_n \pi}{2} \sin \frac{\gamma_n \pi}{2}}{n^2 - s^2}$$

$$d_{ns} = \frac{\alpha_n \cosh \frac{\beta_n \pi}{2} \sin \frac{\alpha_n \pi}{2}}{\alpha_n^2 - s^2} - \frac{\beta_n \cos \frac{\alpha_n \pi}{2} \sinh \frac{\beta_n \pi}{2}}{\beta_n^2 + s^2}.$$

Type 3

$$c_{ns} = \frac{\gamma_n \cos \frac{\gamma_n \pi}{2} \sin \frac{\delta_n \pi}{2} - \delta_n \cos \frac{\delta_n \pi}{2} \sin \frac{\gamma_n \pi}{2}}{n^2 - s^2}$$

$$d_{ns} = \frac{\alpha_n \cos \frac{\beta_n \pi}{2} \sin \frac{\alpha_n \pi}{2}}{\alpha_n^2 - s^2} - \frac{\beta_n \cos \frac{\alpha_n \pi}{2} \sin \frac{\beta_n \pi}{2}}{\beta_n^2 - s^2}.$$

Type 4

$$c_{ns} = \frac{2\left(\gamma_n \cos\frac{\delta_n\pi}{2}\sin\frac{\delta_n\pi}{2} - \delta_n \cosh\frac{\gamma_n\pi}{2}\sinh\frac{\gamma_n\pi}{2}\right)}{n^2 - s^2}$$

$$d_{ns} = \sinh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2}\left\{\frac{\alpha_n\cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2} - (\beta_n - s)\sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}}{\alpha_n^2 - (\beta_n - s)^2}\right.$$
$$\left. + \frac{\alpha_n\cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2} - (\beta_n + s)\sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}}{\alpha_n^2 + (\beta_n + s)^2}\right\}$$
$$- \cosh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2}\left\{\frac{\alpha_n\sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2} + (\beta_n - s)\cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_2\pi}{2}}{\alpha_n^2 + (\beta_n - s)^2}\right.$$
$$\left. + \frac{\alpha_n\sinh\frac{\alpha_n\pi}{2}\cos\frac{\beta_n\pi}{2} + (\beta_n + s)\cosh\frac{\alpha_n\pi}{2}\sin\frac{\beta_n\pi}{2}}{\alpha_n^2 + (\beta_n + s)^2}\right\}.$$

Eliminating the *As* and *Bs* between (17) and (18), the infinite determinant is

$$\begin{vmatrix} a_{12} & b_{12} & a_{32} & b_{32} & \cdot & \cdot \\ d_{10} & c_{10} & d_{30} & b_{30} & \cdot & \cdot \\ a_{14} & b_{14} & a_{34} & b_{34} & \cdot & \cdot \\ d_{12} & c_{12} & d_{32} & c_{32} & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0. \tag{19}$$

The solution expressed by (19) is merely formal. To find out whether it can be used to determine actual values for the buckling loads, we must examine its convergence. For this purpose we may form a series of finite determinants $\Delta_1, \Delta_2, \ldots, \Delta_n$ by taking $2, 4, \ldots, 2N$ rows and columns starting at the left hand top corner of Δ. If it is found that any root of $\Delta_N = 0$ converges to a definite limit as N increases, this root represents a possible condition for which an elastic displacement can exist in neutral equilibrium.

RESULTS

In order to get the buckling coefficient for various loading cases, a computer program was developed which enables the iterations for $\Delta = 0$ to be done. Results are presented to cover the range of aspect ratios from 1.0 to 4.0 for a range of loading cases from the uniaxial to uniform biaxial loadings. The results are presented in the table below.

Figure 2 illustrates the results above. It is clear from the figure that there are transitions from one mode to the next as the aspect ratio gets higher. Furthermore, the transition from one mode to the next shifts to higher aspect ratios when P_2 increases. Also, as the aspect ratio approaches 4.0, the buckling coefficient approaches a horizontal asymptote.

DISCUSSION

The family of curves, shown in Fig. 2, give reasonable values for buckling load for various load ratios, which range from uniaxial to uniform biaxial loading. As expected, varying P_2 from 0.0 to 1.0 while P_1 is constant at 1.0, causes the value of the load at which buckling occurs to decrease. The transition from one mode of buckling to the next, which shifts to higher aspect ratios when P_2 increases, is clearly illustrated. At all load ratios, the buckling coefficient approaches a horizontal asymptote as the aspect ratio increases (beyond or near 4.0). The results for uniaxial loading ($P_1 = 1$, $P_2 = 0$) agree well with those of other investigators. Comparisons are shown in Figs. 3, 5 and 6. As the other results have been scaled from published figures too much weight should not be put on any differences.

Table 1. Buckling coefficient, $K = 4P_1b^2/D^2$ as a function of load and aspect ratios

Aspect ratio	Factor on P_1 and P_2 for buckling, K						
	$P_2 = 0$	$P_2 = 0.1$	$P_2 = 0.2$	$P_2 = 0.3$	$P_2 = 0.5$	$P_2 = 0.7$	$P_2 = 1.0$
1	10.0738	9.3147	8.6382	8.0376	7.0309	6.2296	5.3035
1.1	9.9886	9.1533	8.4048	7.3438	6.6548	5.8114	4.8658
1.2	9.5919	9.1177	8.3160	7.5999	6.4262	5.5394	4.8684
1.3	9.0083	8.6658	8.2982	7.5460	6.2981	5.3631	4.3637
1.4	8.6123	8.2370	7.8885	7.5350	6.2304	5.2505	4.2210
1.5	8.3566	7.9460	7.5675	7.2181	6.2013	5.1807	4.1213
1.6	8.2050	7.7588	7.3486	6.9725	6.1923	5.1394	4.0507
1.7	8.1288	7.6481	7.2068	6.8031	6.1012	5.1167	4.0008
1.8	8.1016	7.5934	7.1225	6.6921	5.9471	5.1062	3.9656
1.9	8.0417	7.5735	7.0795	6.6240	5.8369	5.1021	3.9408
2.0	7.8674	7.5201	7.0629	6.5875	5.7607	5.0916	3.9234
2.1	7.7417	7.3774	7.0333	6.5721	5.7096	5.0130	3.9115
2.2	7.6574	7.2736	6.9146	6.5672	5.6784	4.9546	3.9035
2.3	7.6064	7.2027	6.8266	6.4756	5.6610	4.9125	3.8983
2.4	7.5812	7.1589	6.7650	6.3998	5.6529	4.8825	3.8950
2.5	7.5731	7.1362	6.7257	6.3447	5.6497	4.8622	3.8932
2.6	7.5448	7.1273	6.7034	6.3074	5.6055	4.8492	3.8922
2.7	7.4734	7.1200	6.6933	6.2847	5.5597	4.8411	3.8919
2.8	7.4199	7.0576	6.6904	6.2730	5.5252	4.8371	3.8898
2.9	7.3823	7.0094	6.6548	6.2681	5.5005	4.8346	3.8734
3.0	7.3608	6.9745	6.6102	6.2653	5.4835	4.8229	3.8604
3.1	7.3494	6.9518	6.5770	6.2244	5.4734	4.7978	3.8506
3.2	7.3450	6.9383	6.5538	6.1923	5.4681	4.7779	3.8431
3.3	7.3223	6.9328	6.5405	6.1684	5.4661	4.7629	3.8374
3.4	7.2874	6.9304	6.5312	6.1518	5.4653	4.7524	3.8333
3.5	7.2615	6.8979	6.5271	6.1413	5.4413	4.7447	3.8304
3.6	7.2445	6.8724	6.5186	6.1352	5.4223	4.7398	3.8284
3.7	7.2343	6.8542	6.4935	6.1324	5.4077	4.7370	3.8272
3.8	7.2299	6.8424	6.4740	6.1251	5.3968	4.7357	3.8265
3.9	7.2282	6.8355	6.4602	6.1052	5.3891	4.7353	3.8262
4.0	7.2084	6.8327	6.4513	6.0898	5.3842	4.7301	3.8261

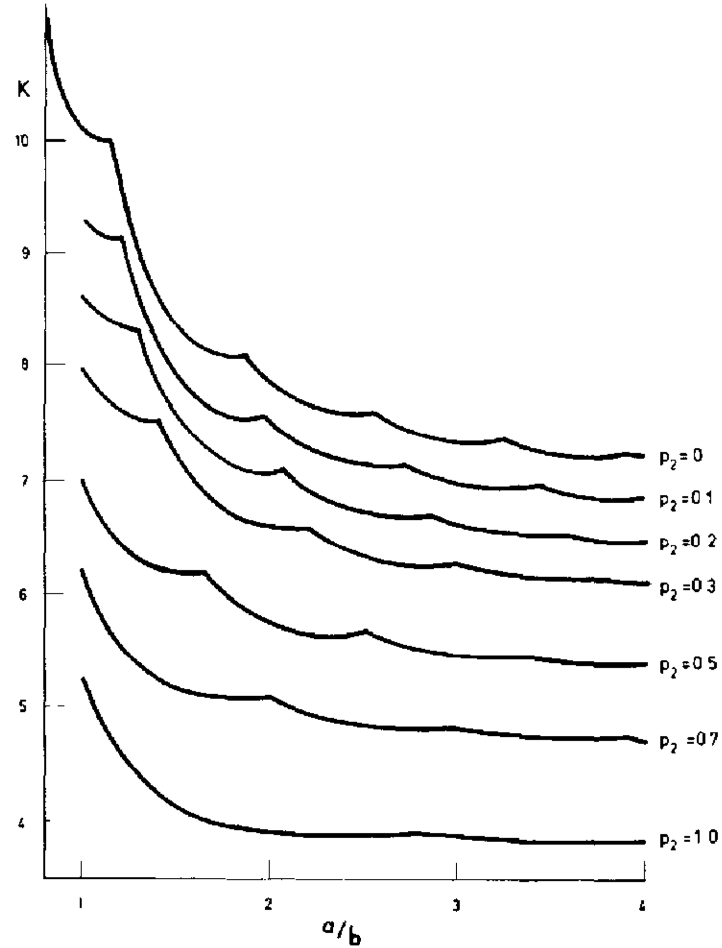

Fig. 2 Buckling coefficient, K, vs aspect ratio for a range of load combinations.

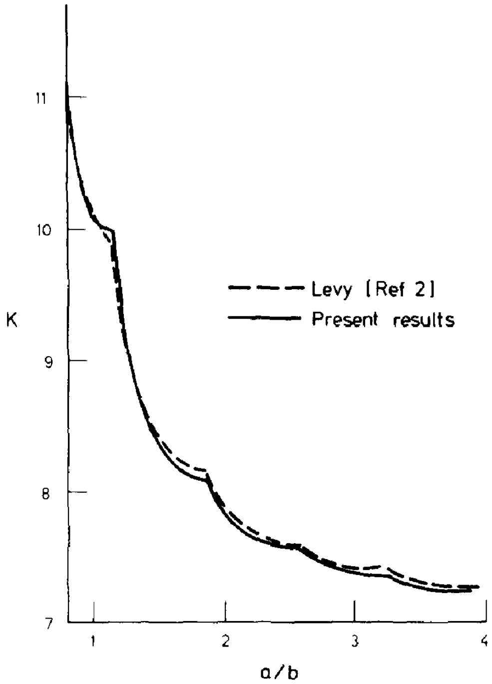

Fig. 3.

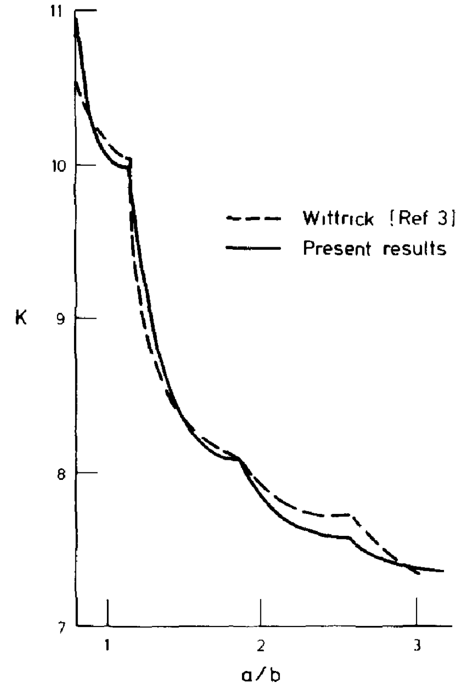

Fig. 4.

Fig. 3. Buckling coefficient, K, for uniaxial loading comparison with Levy, Ref. [2].

Fig. 4. Buckling coefficient, K, for uniaxial loading comparison with Wittrick, Ref. [3].

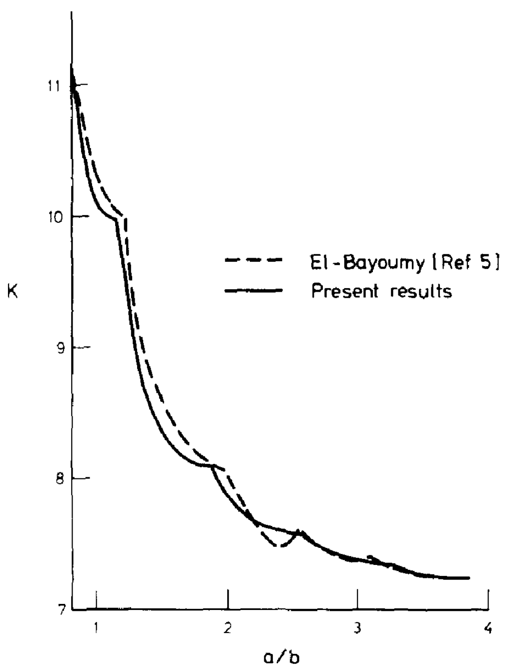

Fig. 5.

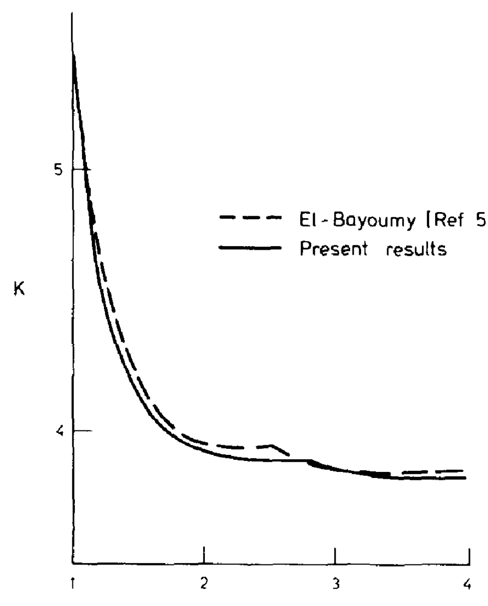

Fig. 6.

Fig. 5. Buckling coefficient, K, for uniaxial loading comparison with El-Bayoumy, Ref. [5].

Fig. 6. Buckling coefficient, K, for biaxial loading comparison with El-Bayoumy, Ref. [5].

In conclusion, this method gives reliable results and is useful because the computer program developed fits into a mini computer and does not require a great deal of storage and computer time.

Completely clamped plates occur relatively rarely in engineering structures. However, when they do occur, for example as thin diaphragms in more rigid structures their buckling behaviour may be important. Also a comparison of the clamped and simply supported buckling loads for a plate can give the designer an indication of the benefit he is likely to gain by increasing edge fixity.

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